

Heterochromatic triangles in edge-colored graphs*

Binlong Li, Bo Ning, Chuandong Xu and Shenggui Zhang[†]

Department of Applied Mathematics, Northwestern Polytechnical University,

Xi'an, Shaanxi 710072, P.R. China

Abstract

Let G be an edge-colored graph. The color degree of a vertex v of G , is defined as the number of colors of the edges incident to v . The color number of G is defined as the number of colors of the edges in G . A heterochromatic triangle is one in which every pair of edges have different colors. In this paper we give some sufficient conditions for the existence of heterochromatic triangles in edge-colored graphs in terms of color degree, color number and edge number. As a corollary, a conjecture proposed by Li and Wang (Color degree and heterochromatic cycles in edge-colored graphs, European J. Combin. 33 (2012) 1958-1964) is confirmed.

Keywords: Edge-colored graphs; Color degree; Color number; Heterochromatic triangles; Directed triangles

1 Introduction

All graphs considered here are simple and finite. For terminology and notation not defined here, we refer to Bondy and Murty [1].

Let $G = (V, E)$ be a graph. We use $e(G)$ to denote the number of edges of G . An *edge-coloring* of G is a mapping $C : E \rightarrow \mathbb{N}^+$, where \mathbb{N}^+ is the set of natural numbers.

*Supported by NSFC (No. 11271300) and the Doctorate Foundation of Northwestern Polytechnical University (No. cx201202)

[†]Corresponding author. E-mail address: sgzhang@nwpu.edu.cn

We call G an *edge-colored graph* (or briefly, a colored graph) if it is assigned such an edge-coloring C . Let v be a vertex of G . The *color degree* of v in G , denoted by $d_G^c(v)$ (or briefly, $d^c(v)$), is defined as the number of colors of the edges incident to v . We use $C(G)$ to denote the set, and $c(G)$ the number, of colors of edges in G . We call $c(G)$ the *color number* of G . A triangle in a colored graph is called *heterochromatic* if every two of its edges have different colors.

In this paper, we mainly study the existence of heterochromatic triangles in colored graphs. The following theorem concerning triangles in (non-colored) graphs is well known.

Theorem 1 (Mantel [4]). *Let G be a graph on n vertices. If $e(G) > \lfloor n^2/4 \rfloor$, then G contains a triangle.*

It is natural to consider that how many colors can guarantee a heterochromatic triangle in a colored graph. The answer is obvious.

Theorem 2. *Let G be a colored graph on n vertices. If $c(G) > \lfloor n^2/4 \rfloor$, then G contains a heterochromatic triangle.*

In fact, we can take $c(G) > \lfloor n^2/4 \rfloor$ edges in G such that every two of them have different colors. By Theorem 1, the edge-induced subgraph on all the taken edges contains a triangle, and this triangle is heterochromatic.

On the other hand, the bound of Theorem 2 is sharp. Note that $G = K_{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor}$ with edges assigned pairwise different colors contains no (heterochromatic) triangles but $c(G) = \lfloor n^2/4 \rfloor$.

Now we give two sufficient conditions for the existence of a heterochromatic triangle in a colored graph.

Theorem 3. *Let G be a colored graph on n vertices. If $e(G) + c(G) \geq n(n+1)/2$, then G contains a heterochromatic triangle.*

Theorem 4. *Let G be a colored graph on n vertices. If $\sum_{v \in V(G)} d^c(v) \geq n(n+1)/2$, then G contains a heterochromatic triangle.*

Let G be a complete graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$. For the edge $v_i v_j$, $1 \leq i < j \leq n$, we assign the color i to it. Then $e(G) + c(G) = \sum_{v \in V(G)} d^c(v) = n(n+1)/2 - 1$, and G contains no heterochromatic triangles. This implies that the bounds of Theorems 3 and 4 are both sharp.

Li and Wang [3] conjectured that a colored graph G on n vertices contains a heterochromatic triangle if $d^c(v) \geq (n+1)/2$ for every vertex $v \in V(G)$. As a corollary of Theorem 4, we can see that Li and Wang's conjecture is true.

Corollary 1. *Let G be a colored graph on n vertices. If $d^c(v) \geq (n+1)/2$ for every vertex $v \in V(G)$, then G contains a heterochromatic triangle.*

With more efforts, we can prove the following stronger theorem.

Theorem 5. *Let G be a colored graph on n vertices. If $d^c(v) \geq n/2$ for every vertex $v \in V(G)$ and G contains no heterochromatic triangles, then G is the bipartite graph $K_{n/2, n/2}$, unless $G = K_4 - e$ or K_4 when $n = 4$.*

Let $D = (V, A)$ be a digraph and v be a vertex of D . We use $N_D^+(v)$ ($N_D^-(v)$) to denote the set of out-neighbors (in-neighbors), and $d_D^+(v)$ ($d_D^-(v)$), the out-degree (in-degree) of v in D . For $S \subset V(D)$, $D[S]$ denotes the subdigraph induced by S . The *out-component number* of v , denoted by $\omega_D^+(v)$, is the number of the components of $D[N^+(v)]$. When no confusion occurs, we use $N^+(v)$, $N^-(v)$, $d^+(v)$, $d^-(v)$ and $\omega^+(v)$ instead of $N_D^+(v)$, $N_D^-(v)$, $d_D^+(v)$, $d_D^-(v)$ and $\omega_D^+(v)$, respectively. An *orientation* of a (undirected) graph G is a digraph obtained from G by replacing each edge by one of the two possible arcs with the same ends. Such a digraph is called an *oriented graph*.

The research of directed triangles is closely related to that of heterochromatic triangles. Let D be an oriented graph. We construct a colored graph as follows: Let v be an arbitrary

vertex of D and H be a component of $D[N^+(v)]$. We assign a same color to the arcs from v to all the vertices in H . For two arcs with different tails, or with the same tail, say v , but with heads in different components of $D[N^+(v)]$, we assign two different colors to them. We call the underlying graph of D with the given coloring the *associated colored graph* of D , and denote it by $G(D)$. One can see that an oriented graph D contains a directed triangle if and only if $G(D)$ contains a heterochromatic triangle. We omit the details (the readers can find the proof in Section 2).

Let G be an associated colored graph of some oriented graph D . Note that the color degree of v in G is equal to $d_D^-(v) + \omega_D^+(v)$. This implies that $\sum_{v \in V(G)} d^c(v) = \sum_{v \in V(D)} (d_D^-(v) + \omega_D^+(v)) = e(G) + c(G)$. This is the reason why we consider the sum of edge number and color number for the existence of heterochromatic triangles in colored graphs.

Now we come back to digraphs. We use $a(D)$ to denote the number of arcs of a digraph D . In the following, we give two theorems concerning directed triangles corresponding to Theorem 4 and 5, respectively.

Theorem 6. *Let D be an oriented graph on n vertices. If $a(D) + \sum_{v \in V(D)} \omega^+(v) \geq n(n+1)/2$, then D contains a directed triangle.*

Theorem 7. *Let D be an oriented graph on n vertices. If $d^-(v) + \omega^+(v) \geq n/2$ for every vertex $v \in V(D)$, then D either contains a directed triangle or is an orientation of $K_{n/2, n/2}$.*

The proofs of Theorems 4 and 5 are heavily based on Theorems 6 and 7, respectively.

The following conjecture concerning directed triangles, which is a special case of the famous Caccetta-Häggkvist Conjecture, is still open.

Conjecture 1 (Caccetta and Häggkvist [2]). *Any oriented graph on n vertices with minimum in-degree at least $n/3$ contains a directed triangle.*

Since this conjecture is difficult to prove, one may seek for the value α as small as

possible such that every oriented graph on n vertices with minimum in-degree at least αn contains a directed triangle. The best value of α known to us is obtained by N. Lichiardopol [5]. We list the following result due to Shen, which supports our proof of Theorem 7.

Theorem 8 (Shen [6]). *If $\alpha = 3 - \sqrt{7} = 0.3542 \dots$, then any oriented graph on n vertices with minimum in-degree at least αn contains a directed triangle.*

2 Proofs of the theorems

Proof of Theorem 3.

Suppose the contrary. Let G be a counterexample with the smallest vertex number, and then with the smallest edge number.

Claim 1. G contains two edges with the same color.

Proof. If every two edges of G have different colors, then $e(G) = c(G) \geq n(n+1)/4 > \lfloor n^2/4 \rfloor$. By Theorem 2, there is a heterochromatic triangle in G , a contradiction. \square

Claim 2. $e(G) + c(G) = n(n+1)/2$.

Proof. By Claim 1, let e_1 and e_2 be two edges with the same color. Then $e(G - e_1) = e(G) - 1$ and $c(G - e_1) = c(G)$. If $e(G) + c(G) \geq n(n+1)/2 + 1$, then $e(G - e_1) + c(G - e_1) \geq n(n+1)/2$. Note that $G - e_1$ does not contain a heterochromatic triangle. Thus $G - e_1$ is a counterexample with fewer edges, a contradiction. \square

Let v be a vertex in G , and s a color in $C(G)$. If all the edges with color s are incident to v , then we call s a color *saturated* by v . We use $d^s(v)$ to denote the number of colors saturated by v .

Claim 3. For every $v \in V(G)$, $d(v) + d^s(v) \geq n + 1$.

Proof. Note that $e(G - v) = e(G) - d(v)$. If a color in $C(G)$ is not saturated by v , then it is also a color in $C(G - v)$. This implies that $c(G - v) = c(G) - d^s(v)$. If $d(v) + d^s(v) \leq n$, then

$$e(G - v) + c(G - v) = e(G) - d(v) + c(G) - d^s(v) \geq n(n - 1)/2.$$

Note that $G - v$ does not contain a heterochromatic triangle. Thus $G - v$ is a counterexample with fewer vertices, a contradiction. \square

Claim 4. $\sum_{v \in V(G)} d^s(v) \leq 2c(G)$, and the equality holds if and only if every two edges have different colors.

Proof. Let c be an arbitrary color in $C(G)$. Note that c cannot be saturated by more than two vertices, and c is saturated by exactly two vertices if and only if c appears on only one edge. Thus we have $\sum_{v \in V(G)} d^s(v) \leq 2c(G)$, and the equality holds if and only if every two edges have different colors. \square

By Claims 2, 3 and 4, we can get that

$$n(n + 1) \leq \sum_{v \in V(G)} (d(v) + d^s(v)) \leq 2e(G) + 2c(G) = n(n + 1).$$

This implies that $\sum_{v \in V(G)} (d(v) + d^s(v)) = 2e(G) + 2c(G)$ and $\sum_{v \in V(G)} d^s(v) = 2c(G)$. By Claim 4, every two edges have different colors, contradicting to Claim 1.

The proof is complete. \square

Proof of Theorem 4.

The proof of this theorem is based on Theorem 6, which will be proved later. Suppose that G contains no heterochromatic triangles. Let G' be a spanning subgraph of G satisfying the condition of Theorem 4 with edge number as small as possible.

Claim 1. For each edge $uv \in E(G')$, one of the following is true:

- (1) $C(uw) \neq C(uv)$ for $w \in N(u) \setminus \{v\}$;
- (2) $C(vw) \neq C(uv)$ for $w \in N(v) \setminus \{u\}$.

Proof. If $C(uw) = C(uv)$ for some $w \in N(u) \setminus \{v\}$, then the removal of the edge uv does not reduce the color degree of u . If $C(wv) = C(uv)$ for some $w \in N(v) \setminus \{u\}$, then the removal of the edge uv does not reduce the color degree of v . Since G' contains the fewest edges, either (1) or (2) holds. \square

Now we give an orientation to G' in such a way: for $uv \in E(G')$, if (1) of Claim 1 holds, then the orientation of the edge is from v to u ; if (2) of Claim 1 holds, then the orientation is from u to v ; if both (1) and (2) hold, then we give the orientation arbitrarily. We denote the resulting oriented graph by D . By the construction of D , we have

Claim 2. If $uv \in A(D)$, then $C(uv)$ is different from the colors of every other arcs incident to v .

Claim 3. Let v be a vertex of D . If $x, y \in N^+(v)$ and $xy \in A(D)$, then $C(vx) = C(vy)$.

Proof. By Claim 2, $C(vx) \neq C(xy)$ and $C(vy) \neq C(xy)$. If $C(vx) \neq C(vy)$, then $vxyv$ is a heterochromatic triangle in G , a contradiction. \square

By applying Claim 3 repeatedly, we can conclude that if H is a component of $D[N^+(v)]$, then the colors of the arcs from v to all vertices in H are the same.

Claim 4. For every vertex $v \in V(D)$, $d^-(v) + \omega^+(v) \geq d_{G'}^c(v)$.

Proof. By Claim 2, every arc with head v has the color different from the colors of the other arcs incident to v . By Claim 3, the arcs from v to the vertices in the same component of $D[N^+(v)]$ have the same color. Hence $d^-(v) + \omega^+(v) \geq d_{G'}^c(v)$. \square

By Claim 4, we have

$$a(D) + \sum_{v \in V(D)} \omega^+(v) = \sum_{v \in V(D)} (d^-(v) + \omega^+(v)) \geq \sum_{v \in V(G')} d_{G'}^c(v) \geq n(n+1)/2.$$

By Theorem 6, there is a directed triangle in D , say $uvwu$. By Claim 2, $C(uw) \neq C(uv)$, $C(uv) \neq C(vw)$ and $C(vw) \neq C(uw)$. Therefore, $uvwu$ is a heterochromatic triangle in G , a contradiction.

The proof is complete. \square

Proof of Theorem 5.

The proof of this theorem is based on Theorem 7. Suppose that G contains no heterochromatic triangles. Let G' be a spanning subgraph of G satisfying the condition of Theorem 5 with edge number as small as possible. As in the proof of Theorem 4, we have

Claim 1. For each edge $uv \in E(G')$, one of the following is true:

- (1) $C(uw) \neq C(uv)$ for $w \in N(u) \setminus \{v\}$;
- (2) $C(wv) \neq C(uv)$ for $w \in N(v) \setminus \{u\}$.

Now we give an orientation to G' as in the proof of Theorem 4, and similarly, we have

Claim 2. If $uv \in A(D)$, then $C(uv)$ is different from the colors of every other arcs incident to v .

Claim 3. Let v be a vertex of D . If $x, y \in N^+(v)$ and $xy \in A(D)$, then $C(vx) = C(vy)$.

Claim 4. For every vertex $v \in V(D)$, $d^-(v) + \omega^+(v) \geq d_{G'}^c(v)$.

By Claim 4, we have $d^-(v) + \omega^+(v) \geq n/2$ for every $v \in V(D)$. By Theorem 7, there is either a directed triangle in D or D is an orientation of the complete bipartite graph $K_{n/2, n/2}$. If there is a directed triangle in D , then it is a heterochromatic triangle in G , a contradiction. Thus we assume that D is an orientation of $G' = K_{n/2, n/2}$, where n is even.

For any vertex $v \in V(G')$, since $d_{G'}^c(v) \geq n/2$ and $d_{G'}(v) = n/2$, every pair of edges incident to v have different colors. Note that G is a spanning supergraph of G' . If $n = 2$, then $G = K_2$. If $n = 4$, then $G = K_{2,2}, K_4 - e$ or K_4 . Now suppose that $n \geq 6$, and we will show that there are no edges in $E(G) \setminus E(G')$. If not, then we assume that $uv \in E(G)$ with u, v in a same partition set of the bipartite graph G' . Let x, y, z be three vertices in the other partition set of G' . Since ux, uy and uz have pairwise different colors, there are at least two edges in $\{ux, uy, uz\}$ with colors different from uv . Similarly, there are at

least two edges in $\{vx, vy, vz\}$ with colors different from uv . Hence either $uvxu$, $uvyu$, or $uvzu$ is a heterochromatic triangle in G , a contradiction.

The proof is complete. \square

Proof of Theorem 6.

Let G be the associated colored graph of D . We first prove the following claim.

Claim 1. D has a directed triangle if and only if G has a heterochromatic triangle.

Proof. If D has a directed triangle, say $uvwu$, then by the definition of associated colored graphs, $C(uv) \neq C(vw)$, $C(vw) \neq C(wu)$ and $C(wu) \neq C(uv)$. Thus $uvwu$ is a heterochromatic triangle in G .

Conversely, suppose that G contains a heterochromatic triangle, say $uvwu$. If $\{u, v, w\}$ does not induce a directed triangle, then there is a vertex, say u , dominating the other two vertices. But in this case, v and w are in the same component of $D[N^+(u)]$. By the definition of associated colored graphs, $C(uv) = C(uw)$, a contradiction. \square

Note that $e(G) = a(D)$ and $c(G) = \sum_{v \in V(D)} \omega^+(v)$. We have

$$e(G) + c(G) = a(D) + \sum_{v \in V(D)} \omega^+(v) \geq n(n+1)/2.$$

By Theorem 3, there is a heterochromatic triangle in G ; and by Claim 1, there is a directed triangle in D .

The proof is complete. \square

Proof of Theorem 7.

We prove the theorem by induction on n . Since the result is trivially true when $n = 2, 3$, we assume that $n \geq 4$. If $d^-(v) \geq \alpha n$ for every vertex $v \in V(D)$, then by Theorem 8, there is a directed triangle. Thus we suppose that there is a vertex v such that

$$d^-(v) < \alpha n. \tag{1}$$

Noting that $d^-(v) + \omega^+(v) \geq n/2$, we have

$$\omega^+(v) \geq n/2 - d^-(v). \quad (2)$$

Claim 1. There is a component of $D[N^+(v)]$ with only one vertex.

Proof. We use $b(v)$ to denote the number of vertices which are not adjacent to v .

Suppose that every component of $D[N^+(v)]$ has at least two vertices. Then

$$n = d^-(v) + d^+(v) + 1 + b(v) \geq d^-(v) + 2\omega^+(v) + 1 + b(v),$$

and by (2),

$$b(v) \leq n - d^-(v) - 2\omega^+(v) - 1 \leq n - d^-(v) - 2(n/2 - d^-(v)) - 1.$$

That is,

$$b(v) \leq d^-(v) - 1. \quad (3)$$

Let H be the subdigraph of D induced by $N^-(v)$. If for every vertex $u \in V(H)$, $d_H^-(u) \geq \alpha d^-(v) = \alpha|V(H)|$, then by Theorem 8, there is a directed triangle in H . Thus we assume that there is a vertex $u \in V(H)$ such that $d_H^-(u) < \alpha d^-(v)$.

First for every $w \in N^+(v)$, $wu \notin A(D)$; otherwise $uvwu$ is a directed triangle. Since $uv \in A(D)$, all the out-neighbors of u in $\{v\} \cup N^-(v) \cup N^+(v)$ are in the same component of $D[N^+(u)]$. For every vertex nonadjacent to v , it contributes at most one to $d^-(u) + \omega^+(u)$. Thus we have

$$d^-(u) + \omega^+(u) \leq d_H^-(u) + 1 + b(v) < \alpha d^-(v) + 1 + b(v).$$

Since $d^-(u) + \omega^+(u) \geq n/2$, we have

$$b(v) > n/2 - 1 - \alpha d^-(v). \quad (4)$$

Combining (3) with (4), we have $n/2 - 1 - \alpha d^-(v) < d^-(v) - 1$, and

$$d^-(v) > \frac{n}{2(1+\alpha)} > \alpha n$$

(noting that $2\alpha(1+\alpha) = 0.9594 \dots < 1$), contradicting to (1). \square

Now let w be an isolated vertex of $D[N^+(v)]$, and let $D' = D - \{v, w\}$.

Claim 2. For every vertex $u \in V(D')$, $d_{D'}^-(u) + \omega_{D'}^+(u) \geq d^-(u) + \omega^+(u) - 1$.

Proof. First we assume that $u \in N^-(v)$. Note that $wu \notin A(D)$; otherwise $uvwu$ will be a directed triangle. We have $d_{D'}^-(u) = d^-(u)$. If $uw \in A(D)$, then v and w are in the same component of $D[N^+(u)]$. Since the removal of $\{v, w\}$ does not change the components of $D[N^+(u)]$ not containing v , we have $\omega_{D'}^+(u) \geq \omega^+(u) - 1$, and then $d_{D'}^-(u) + \omega_{D'}^+(u) \geq d^-(u) + \omega^+(u) - 1$.

Next we assume that $u \in N^+(v) \setminus \{w\}$. Since w is an isolated vertex of $D[N^+(v)]$, it is not adjacent to u . This implies that $d_{D'}^-(u) = d^-(u) - 1$ and $\omega_{D'}^+(u) = \omega^+(u)$. Thus $d_{D'}^-(u) + \omega_{D'}^+(u) = d^-(u) + \omega^+(u) - 1$.

At last, we assume that u is not adjacent to v . If u and w are not adjacent to each other, then the removal of $\{v, w\}$ does not change the in- and out-neighbors of u . If $wu \in A(D)$, then $d_{D'}^-(u) = d^-(u) - 1$ and $\omega_{D'}^+(u) = \omega^+(u)$. If $uw \in A(D)$, then $d_{D'}^-(u) = d^-(u)$, and the removal of $\{v, w\}$ does not change the components of $D[N^+(u)]$ not containing w . In any case, we have $d_{D'}^-(u) + \omega_{D'}^+(u) \geq d^-(u) + \omega^+(u) - 1$. \square

By induction hypothesis, D' contains a directed triangle or is an orientation of $K_{n/2-1, n/2-1}$.

If D' contains a directed triangle, then it is also a directed triangle in D . Now we assume that D' is an orientation of $K_{n/2-1, n/2-1}$, where n is even. Let $V(D') = X \cup Y$, where X and Y are two partition sets of the bipartite graph D' .

Claim 3. For every vertex $u \in V(D) \setminus \{v, w\}$, u is adjacent to exactly one vertex of $\{v, w\}$.

Proof. If u is not adjacent to both v and w , then $d^-(u) + \omega^+(u) \leq d^-(u) + d^+(u) = n/2 - 1$, a contradiction. This implies that any vertex in $V(D) \setminus \{v, w\}$ is adjacent to at least one vertex in $\{v, w\}$.

Now suppose the contrary that u is adjacent to both v and w . If $vu \in A(D)$, then w and u are in the same component of $D[N^+(v)]$, contradicting to that w is an isolated

vertex of $D[N^+(v)]$. Thus we assume that $uv \in A(D)$. If $wu \in A(D)$, then $uvwu$ is a directed triangle. Thus we assume that $uw \in A(D)$. Without loss of generality, we assume that $u \in X$.

Let $y \in Y$. We claim that $yu \in A(D)$. Suppose the contrary that $uy \in A(D)$. Since y is adjacent to either v or w , $\{y, v, w\}$ is contained in a same component of $D[N^+(u)]$. Note that u is adjacent to $n/2 - 2$ vertices other than y, v and w . This implies that $d^-(u) + \omega^+(u) \leq n/2 - 1$, a contradiction. Thus as we claimed, $yu \in A(D)$.

If $vy \in A(D)$ or $wy \in A(D)$, then $uvyu$ or $uwyu$ is a directed triangle. Thus we assume that $vy \notin A(D)$ and $wy \notin A(D)$. Note that v, w (if dominated by y) and u are in the same component of $D[N^+(y)]$, and y is adjacent to $n/2 - 2$ vertices other than u, v and w . This implies that $d^-(y) + \omega^+(y) \leq n/2 - 1$, a contradiction. \square

Since $d^-(v) + d^+(v) \geq d^-(v) + \omega^+(v) \geq n/2$ and $d^-(w) + d^+(w) \geq d^-(w) + \omega^+(w) \geq n/2$, by Claim 3, we can see that $d^-(v) + d^+(v) = n/2$, $d^-(w) + d^+(w) = n/2$. This implies that every vertex in D is adjacent to exactly $n/2$ vertices. We claim that for every $u \in V(D)$, $N^+(u)$ is an independent set. If not, then there is a component of $D[N^+(u)]$ containing at least two vertices. This implies that $\omega^+(u) < d^+(u)$ and $d^-(u) + \omega^+(u) < n/2$, a contradiction.

Now we claim that v cannot be adjacent to one vertex $x \in X$ and one vertex $y \in Y$. Suppose not. If $\{x, y, v\}$ does not induce a directed triangle, then there is a vertex, say x , dominating the other two vertices. But in this case, $N^+(x)$ is not an independent set, a contradiction.

Without loss of generality, we assume that v is not adjacent to any vertex in Y and then adjacent to all the vertices in X . By Claim 3, w is not adjacent to any vertex in X and adjacent to all the vertices in Y . Thus D is an orientation of $K_{n/2, n/2}$.

The proof is complete. \square

References

- [1] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications, Macmillan London and Elsevier, New York (1976).
- [2] L. Caccetta and R. Häggkvist, On minimal digraphs with given girth, in: Proceedings, Ninth S-E Conference on Combinatorics, Graph Theory and Computing, (1978) 181-187.
- [3] H. Li and G. Wang, Color degree and heterochromatic cycles in edge-colored graphs, *European J. of Combin.* **33** (2012) 1958-1964.
- [4] W. Mantel, Problem 28, Wiskundige Opgaven **10** (1907) 60-61.
- [5] N. Lichiardopol, A new bound for a particular case of the Caccetta-Häggkvist conjecture, *Discrete Math.* **310** (23) (2010) 3368-3372.
- [6] J. Shen, Directed triangles in digraphs, *J. Combin. Theory B* **74** (1998) 405-407.